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Derivative Bounds for Müntz Polynomials*

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Let Λ be a finite set of positive numbers, λ, and consider expressions of the form $C_0 + \sum_\Lambda C_\lambda x^\lambda$. We wish to detremine just how much such a function can be magnified by differentiation on [0, 1]. Since 0 is a peculiar point (where x^λ can even have an infinite derivative), we find it more natural to consider the operator $x(d/dx)$. Equivalently we may transform variables by $x = e^{-t}$ which reduces our operator to d/dt and our interval to $[0, \infty)$.

So call the expressions $P(t) = C_0 + \sum_\Lambda C_\lambda e^{-\lambda t}$ Λ-polynomials and introduce the norm $\| f(t) \| = \mathrm{Sup}_{[0,\infty)} | f(t) |$.

Our problem, in precise terms, is to find $\| d/dt \|_\Lambda = \mathrm{Sup}_P \| P'(t) \| / \| P(t) \|$ taken over all nontrivial Λ-polynomials.

For example, when Λ consists of the first n integers, then this reduces to the classical Tchebychev polynomial case and the answer is known to be $2n^2$. One corollary of our present results is that if Λ consists of *any* n distinct integers, then $\| d/dt \|_\Lambda \geqslant \frac{1}{3} n^2$. (Perhaps this is even true with $2n^2$ replacing $\frac{1}{3}n^2$, but we cannot obtain this precision.)

THEOREM. *For every set Λ we have*

$$\tfrac{2}{3} \sum_\Lambda \lambda \leqslant \| d/dt \|_\Lambda \leqslant 11 \sum_\Lambda \lambda.$$

Proof. We may assume, w.l.o.g., that $\sum_\Lambda \lambda = 1$ as this can always be achieved by a mere change of scale, $t' = ct$. So set $B(z) = \prod_\Lambda ((z - \lambda)/(z + \lambda))$ and define

$$T(t) = \frac{1}{2\pi i} \int_\Gamma \frac{e^{-zt}}{B(z)}\, dz, \qquad \Gamma \text{ the circle } | z - 1 | = 1. \tag{1}$$

A direct application of the residue theorem shows that $T(t)$ is indeed a Λ-polynomial.

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To obtain our desired estimates we will need the following

LEMMA. *All along Γ, $|B(z)| \geqslant \frac{1}{3}$.*

Proof. The function $(z-\lambda)/(z+\lambda)$ maps Γ onto the circle whose diameter is the interval $[-1, (2-\lambda)/(2+\lambda)]$ so that, on Γ, we certainly have

$$\left|\frac{z-\lambda}{z+\lambda}\right| \geqslant \frac{2-\lambda}{2+\lambda} = \frac{1-(\lambda/2)}{1+(\lambda/2)}$$

and therefore

$$|B(z)| \geqslant \prod \frac{1-(\lambda/2)}{1+(\lambda/2)}.$$

To estimate this product, note that, for x and $y \geqslant 0$,

$$\frac{1-x}{1+x}\,\frac{1-y}{1+y} = \frac{1-(x+y)}{1+x+y} + \frac{2xy}{(1+x)(1+y)(1+xy)} \geqslant \frac{1-(x+y)}{1+x+y},$$

and this inequality used repeatedly gives

$$\prod \frac{1-(\lambda/2)}{1+(\lambda/2)} \geqslant \frac{1-\frac{1}{2}\sum\lambda}{1+\frac{1}{2}\sum\lambda} = \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3},$$

as required.

We can now prove our lower bound. For the Λ-polynomial $T(t)$ we have by (1) and our lemma,

$$|T(t)| \leqslant \frac{1}{2\pi}\int_\Gamma \left|\frac{e^{-zt}}{B(z)}\right| |dz| \leqslant \frac{1}{2\pi}\cdot 3\cdot 2\pi = 3, \tag{2}$$

while, again by (1), $T'(0) = -(1/2\pi i)\int_\Gamma (z/B(z))\,dz$ and this integral can be evaluated by taking the residue at ∞, there being no poles outside of Γ. We have, namely,

$$\frac{1}{B(z)} = \prod \frac{1+(\lambda/z)}{1-(\lambda/z)} = 1 + \frac{2\sum\lambda}{z} + \frac{2(\sum\lambda)^2}{z^2} + \cdots$$

$$= 1 + \frac{2}{z} + \frac{2}{z^2} + \cdots \tag{3}$$

so that $z/B(z) = z + 2 + (2/z) + \cdots$, the residue is 2, and we obtain

$$T'(0) = -2. \tag{4}$$

Our lower bound is implied by (2) together with (4).

Our upper bound is more complicated and we begin with an estimate for $\int_0^\infty |T''(t)|\,dt$. To this end we see from (1) that

$$T''(t) = (1/2\pi i) \int_\Gamma (z^2 e^{-zt}/B(z))\,dz, \tag{5}$$

so that, writing $z = 1 + e^{i\theta}$ and applying our lemma, we obtain $|T''(t)| \leqslant (3/2\pi) \int_0^{2\pi} 2(1 + \cos\theta)\, e^{-(1+\cos\theta)t}\,d\theta$, which yields

$$\int_0^\infty |T''(t)|\,dt \leqslant \frac{3}{2\pi} \int_0^{2\pi} 2(1 + \cos\theta) \frac{1}{1 + \cos\theta}\,d\theta = 6. \tag{6}$$

Next we evaluate $\int_0^\infty e^{-\lambda t} T''(t)\,dt$, $\lambda \in \Lambda$. From (5) we have $\int_0^\infty e^{-\lambda t} T''(t)\,dt = (1/2\pi i) \int_\Gamma (z^2\,dz/B(z)(z + \lambda))$ and as before there are no poles outside of Γ and we need only find the residue at ∞. Again, by (3), we have $1/B(z) = 1 + (2/z) + (2/z^2) + \cdots$ and we also have $1/(z + \lambda) = (1/z) - (\lambda/z^2) + (\lambda^2/\lambda^3) \cdots$ so that combining gives $z^2/B(z)(z + \lambda) = z + 2 - \lambda + ((\lambda^2 - 2\lambda + 2)/z) + \cdots$. Thus $\int_0^\infty e^{-\lambda t} T''(t)\,dt = \lambda^2 - 2\lambda + 2$, and taking linear combinations results in

$$\int_0^\infty P(t + a)\, T''(t)\,dt = P''(a) - 2P'(a) + 2P(a) \tag{7}$$

for any Λ-polynomial.

Applying our estimate (6) to (7), then, gives the bound $|P''(a)| \leqslant 2\,|P'(a)| + 8\,\|P(t)\|$ and letting a vary yields

$$\|P''\| \leqslant 2\,\|P'\| + 8\,\|P\| \tag{8}$$

It is a well-known result, however (see [1]), that for any C^2 function on $[0, \infty)$, $\|f'\|^2 \leqslant 4\|f\| \cdot \|f''\|$. If this is applied to (8) we obtain $\frac{1}{4}(\|P'\|/\|P\|)^2 \leqslant 2(\|P'\|/\|P\|) + 8$ which trivially ensures $\|P'\|/\|P\| \leqslant 11$, our upper bound.

(It might seem circuitous to have to resort to *second*-derivative estimates, but the direct approach, via $\int_0^\infty |T'(t)|\,dt$, does not appear to lend itself to uniform estimates.)

Reference

1. A. Kolmogorov, Inequalities on successive derivatives..., *C. R. Acad. Sci. Paris* **207** (1938), 764–765.